ON THE ROLE OF LOOKING BACK AT PROVING PROCESSES IN SCHOOL MATHEMATICS: FOCUSING ON ARGUMENTATION

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The purpose of this paper is to theoretically examine the role of looking back at proof-planning processes in school mathematics from the viewpoint of argumentation. Various researches have been elaborating what students should obtain through proof and proving. Among them, particular focus of this paper is on proving processes which are not explicit in proof as a product. This paper illustrate that by looking back at such processes students can obtain two kinds of problems: what the necessary condition of the statement is and whether a statement is true which they could not prove at once. They can also solve them in two specific ways to proving.

Keywords: proving, looking back, proof-planning, argumentation, changing conditions

INTRODUCTION

Proof and proving should be central to students' school mathematical experience (Yackel & Hanna, 2003). In the literature, many researches have been elaborating the question: what students should/can obtain through proof and proving. Among them, there are studies which focus on various roles and function of proof theoretically (de Villiers, 1990; Hanna, 1990) or empirically (Miyazaki, 2000). These studies have pointed out, in addition to verification of truth of a statement, what roles and functions proof can/should play or how we can apply such functions in school mathematics.

On the other hand, in recent years, Hanna & Barbeau (2008) focus on another role of proof, based on Rav (1999), as conveying mathematical methods, tools strategies and concepts for solving problems. Their discussion is based on the reflection on the literature on proof in mathematics education that they "seem to have dealt primarily with the logical aspects of proof and with the problems encountered in having students follow deductive arguments" (Hanna & Barbeau, p.347). Then, by using examples commonly treated in secondary school mathematics curriculum, they discuss that it is possible for students to obtain and utilize such methods etc.

In comparison with the discussion on the functions of proof, the focus of Hanna & Barbeau is shifted to proving as a process, rather than proof as a product. For example, they illustrate that students can obtain the technique related to completing the square through proving truth of the formula for the solution of a quadratic equation. They describe an expected process of proving in which students try to find easy cases of quadratic formula and solve them, then try to find out what makes it easy to solve in such cases. Hanna & Barbeau mention that this technique of completing the square "does not stem logically from a previous statement or axiom" but "is a topic-specific move" and useful for their ongoing mathematical learning (ibid, p.349). Thus, they refer to techniques or methods which appear in a proving process.

A key challenge in mathematics education is to theoretically point out what students can obtain through proving processes. With regard to this question, whatever the answer is, if students obtain anything through the processes, they need at least to clarify what they have done in the processes. In relation to this act of clarifying, Polya (1957/2004) refers to "looking back", which involves not only checking errors or insufficiency of own activities during a process or its result, but also finding and solving new problems based on the result or the process (pp.14-16). Thus, Polya discusses various aspects of looking back. In addition, he gives and summarizes important questions and suggestions on a series of processes of mathematical problem solving. So it is expected that, based on Polya's discussion, we can analyze in detail what roles looking back plays in school mathematics.

Among several aspects of looking back, on checking errors or insufficiency and on finding and solving new problems based on the result of proof, their roles have been already discussed in the studies on "coordination" by Heinze et al. (2008) and the functions of proof respectively. On the other hand, this paper focuses on the role of finding and solving new problems based on processes of proving. Moreover, this paper focuses the following theoretical research questions: "by looking back at proving processes, what kind of new problems can students find and how can they solve them?", "what is a value of the finding and solving problems in school mathematics?".

LOOKING BACK AT PROOF-PLANNING PROCESSES

Looking back at proving processes

This study regards proving as a process of solving a problem to prove. According to Polya (1957/2004), four phases of proving are placed: problem-understanding, proof-planning, proof-constructing, and looking back. "Problem" is a "problem to prove", that is, "to answer the question: is this assertion true or false?" (p.154). Thus, problem-understanding involves clarifying the condition and conclusion of a statement or assertion, grasping its meaning, checking if it seems to be true or not.

Proof-planning is to find deductive connections between the statement and other theorems which one already knows, in order to establish truth (or false) of the statement. Nevertheless, in a process of proof-planning, one does not necessarily think only deductively, but one can think, for example, inductively or abductively. Proof-constructing is to establish truth (or false) of the statement by a connected sequence of deductive reasoning.

As mentioned above, in the case of a problem to prove, looking back consists of: checking errors or insufficiency of own activities during a process or its result; finding and solving new problems based on the result or the process (Polya, 1957/2004, pp.14-16). Except for finding and solving new problems based on the process, lots of literature discusses the role of looking back, whether they use the term of looking back or not (for example, de Villiers, 1990; Hanna, 1990). The author has also analyzed how examining proofs made Greek mathematicians and annotators possible not only to find insufficiency of proofs but also to apply ideas of the proof for supplementary proofs. This kind of looking back can be considered as utilizing weaker version of the "explanation" and "discovery" functions of proof as a product (Tsujiyama, 2010).

These literature, including the author's, does not focus on the role of looking back at processes which are not necessarily explicit in the final product of proof. As Shimizu (1994) illustrates by an expected process, especially proof-planning seems to involve mathematically important ideas which are not become explicit. Also Thurston (1994) describes that even mathematicians engage in proving informally, that is, in a way different from the product, and these processes promote their understanding of proof and their ongoing works. Thus, this paper focuses on looking back at proof-planning processes.

Ways of proof-planning

Among many questions and suggestions which Polya (1957/2004) points out to be useful for devising a plan, main questions are the following two: "do you know a related problem?" and "could you use it?" (p.9). Since there are so many problems related to the problem to prove, Polya gives suggestions to find out related problems which seem to be particularly useful. In the case of a problem to prove, Polya gives a suggestion to focus on the conclusion and replaces the former question of the above: "try to think of a familiar theorem having the same or a similar conclusion" (p.26).

It is extensively pointed out that this kind of backward reasoning from conclusion is useful way of proof-planning in school mathematics as well as the opposite forward reasoning from condition. In recent years, Heinze et al. (2008) analyze complex processes of constructing multi-steps proof from the logical viewpoint. They focus on how to deal with intermediary conditions/conclusions obtained by backward/forward reasoning and then point out an essential aspect of proof competence of "coordination" (p.445). According to their studies, this kind of looking back during a proof-planning process plays a role of controlling to select proper deduction.

When backward and forward reasoning do not work well, Polya gives suggestions to find special, generalize or analogous problems and "try to solve first some related problem" (Polya, 1957/2004, p.10). Such proof-planning by changing the condition is important from the following two reasons. Firstly, in the case of proving a general proposition, students need to draw some diagrams of the proposition in problem-understanding phase. For example, when they understand a problem to prove a statement α: "diagonals in a parallelogram intersect at each midpoint", they need to draw figures and check if the statement seems to be true or not. Through such activities, they can spontaneously find special cases such as rhombus, rectangle or square. So students have an opportunity to consider on special cases.

The second reason is that changing the condition is considered to be an effective way to find simple problem in mathematics. Particularly, with regard to specification, Polya

(1954) refers to using “a leading special case”, that is, specifying a proposed problem to an easier case to solve and then utilize the solution for solving the proposed problem (p.25). This way of proof-planning is seen in several articles but its value has been overlooked in mathematics education research on proof and proving.

For example, Hanna & Barbeau (2008) illustrate that students can obtain the technique related to completing the square through proving the formula for the solution of a quadratic equation $ax^2 + bx + c = 0$. In the illustration, they present an expected process of proving in which students try to find some specific quadratic equations that are easy to be solved and find $x^2 = k$ where k is positive. Then the students consider what makes it easy to solve $x^2 = k$ and find out it is the absence of the linear term (pp.348-349). Based on this expected example of a process, Hanna & Barbeau discuss that the technique of completing the square which students possibly obtain through the process “does not stem logically from a previous statement or axiom” (ibid, p.349). From the interest of this paper, it is also remarkable that $x^2 = k$ or its solution is not explicitly stated in the proof. However, in their discussion part, Hanna & Barbeau do not analyze the role of specification to $x^2 = k$, moreover, they do not even refer to it. On the other hand, according to Polya’s notion of a leading special case, this very way of specifying the problem into an easy one seems to play a crucial role in the proving process.

In addition to backward/forward reasoning and changing the condition, especially in the case of geometry, students can use a diagram for proof-planning. For example, Douek (1999) considers reference used in proof and proving as involving not only mathematical theory but also diagrams or visual evidences and so on. She points out empirically that semantically routed arguments using diagrams or numerical examples play an important role in proving.

Thus, we focus on three ways of proof-planning: backward/forward reasoning; changing the condition; referring to diagrams.

A PROOF-PLANNING PROCESS BASED ON ARGUMENTATION

Students, who are the subjects of proving, do not know how the final product is. Therefore, in analyzing proof-planning processes by those three ways, it is necessary to consider the ways have plausibility. That is: an intermediary condition/conclusion obtained by backward/forward reasoning is not necessarily used in the proof; reasoning obtained under the changed condition is not necessarily applicable under the original condition; an element obtained with reference to a diagram does not necessarily hold generally. Moreover, this paper focuses on proof-planning processes which are not explicit in the final product of proof. Thus, this paper focuses on the concept of argumentation, that is, a process of making and examining an argument or a sequence of arguments or a product of the process. An argument is logically connected (but not necessarily deductive or formal) reasoning.

In this section, firstly, we briefly overview the literature on proof/proving and argumentation. Secondly, we characterize proof-planning processes from the nature of argumentation mainly based on Toulmin (1958/2003). Thirdly, based on the characteristics, we illustrate a proof-planning process. Then, in the next section, we discuss the role of looking back at the process.

Literature on proof/proving and argumentation

Based on mathematicians activities such as Lakatos (1976) or Thurston (1994) described, studies on proof/proving and argumentation mainly focuses on how students can (or cannot) construct a proof by utilizing argumentative activities. Among them, several studies point out connections or continuities between argumentation and proof on the one hand (e.g. Douek, 1999), other studies point out epistemological or cognitive gap or distances between them (e.g. Balacheff, 1991). Based on the findings of these studies, Pedemonte (2007) focuses not only on "contents" but also "structure" and introduces Toulmin's (1958/2003) "layout of arguments" for a methodological tool to analyse relationships between them.

These studies focus on how students can construct a mathematical proof. For the aim of this paper, we can obtain lots of implications for processes of proof-planning but not so many for the role of looking back. Therefore, this paper examines the characteristics of proof-planning from the point of view of argumentation based on the findings of these studies and also the nature of argumentation.

Characteristics of proof-planning from the viewpoint of argumentation

Making an argument based on observations of the conclusion

In the literature on proof/proving and argumentation, they mainly analyze its contents (e.g. reference to diagrams) and structure (e.g. abductive) (Pedemonte, 2007). What is common to both is emphasis on observations to make a conjecture or to confirm whether a conclusion holds or not. For example, Pedemonte (2007) shows the role of "abductive argumentation", which "allows the construction of a claim starting from an observed fact" (p.29) and some difficulties related to it. Such observations of the conclusion is also emphasizes in Toulmin (1958/2003). Actually, Toulmin regard that an assertion to be claimed comes first. Thus, it is one of nature of argumentation to make an argument based on observations of the conclusion. On the other hand, in proving process, proposed statement might be false. Since in the literature foci are on processes of justifying a conjectured "fact", this nature is only for the case that the conclusion holds.

Therefore, as the first character of proof-planning processes, it is pointed out: (i) based on observations of the conclusion, if one confirms that it holds, to make an argument.

Uncertainty

Toulmin's "layout of arguments" consists of six related components of an argument: three elements "Data", "Warrant", and "Claim" which have similar function to minor premise, major premise, and conclusion in syllogism; and three more elements "Backing", "Rebuttal", and "modal Qualifier" which are necessary for ordinary arguments (Figure 1 uses abbreviations by their initials). Since it can capture logical and plausible aspects at the same time, it is an effective tool for the aim of this paper.

Among them, modal qualifier functions to show uncertainty of an argument. This element is necessary since we sometimes have to make an argument by limited data or warrants (Toulmin, 1958/2003, pp.93-94).

Therefore, the second character is: (ii) to make an argument with showing its uncertainty.

Adding the condition

Rebuttals represents “conditions of exception” and are necessary for making an argument with explicating the exceptional cases in which data and warrant cannot implicate a claim (ibid, p.93). Since these exceptional cases are specific cases, it is necessary to check specific cases in which a claim does not hold. This element is necessary because even certain data does not necessarily implicate a claim without suitable bridging warrant.

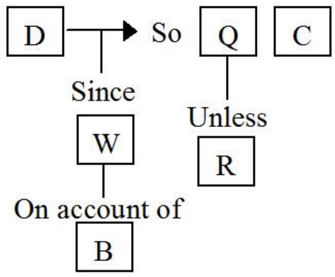

Figure 1 Layout of arguments (Toulmin, 1958/2003)

Therefore, the third and fourth character are pointed out: (iii) based on checking specific cases in which a claim does not hold, if one finds such a case, to make an argument with clarifying what condition is necessary to be added. Since (iii) needs to explicate the condition under consideration, in the rest of the paper, “unless” in Figure 1 is replaced by “under the condition”.

A proof-planning process

To illustrate a proof-planning process, we use the statement α: “diagonals in a parallelogram intersect at each midpoint”. We omit problem-understanding and assume that students translated statement α to Figure 2 and have left OB = OD later.

In Japan, 8th graders are intended to engage in proving this statement at early stage of learning proof/proving. Students have already proved statements such as “opposite sides in a parallelogram are equal” or “opposite angles in a parallelogram are equal” based on the property of parallel lines, alternate angles and the congruent conditions of triangles. As well as these statements, in proof-planning of statement α’, the key idea is to find congruent triangles. If students use backward reasoning appropriately, they can find pair of congruent triangles ADO and CBO or ABO and CDO. However, it is exclusively reported that many Japanese 8th graders have difficulties to find out congruent triangles in such problems. Difficulties seem to be related to the lack of students’ experience of working with such complex diagrams.

Let O be the intersection point of diagonals in a parallelogram ABCD.
Then OA = OC

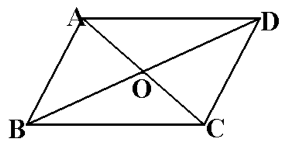

Figure 2 Statement α’

Thus, we assume that students could not find appropriate pair of triangles by backward/forward reasoning and by referring to the diagram and then try to change the condition. Then students will consider in the case of rhombus, rectangle, square, trapezoid and so on. In some cases the conclusion holds but in other cases it does not. In the following, we illustrate a process separately in two kinds of cases.

On the case in which the conclusion holds

If students consider rhombus, by referring to a diagram or also by backward or forward reasoning, they can easily see lots of pairs of congruent triangles. Among these triangles, they may focus triangles ABO and CBO. If they use backward reasoning, they will find that none of the congruent conditions SAS, ASA, SSS can be applied. Then they will find it difficult to obtain a proof. However, based on the character (i), even though they can not prove, they can still make a plausible argument.

In this case, since students have checked all the congruent conditions, it is uncertain whether triangles ABO and CBO are congruent. Therefore, based on the character (ii), they can make an argument like Figure 4.

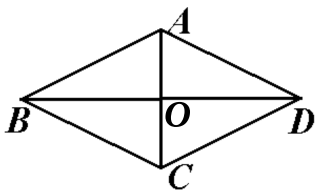

Figure 3 Diagram in the case of a rhombus

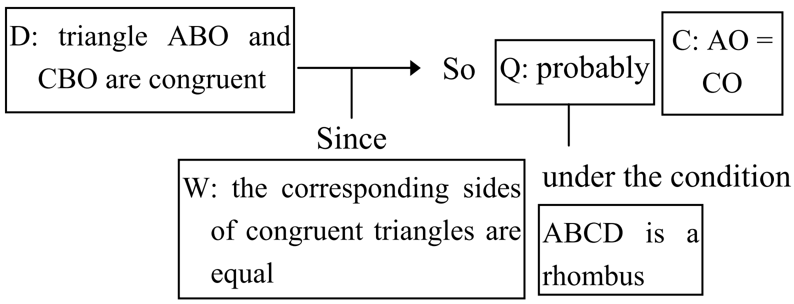

Figure 4 Argument in the case of a rhombus

By changing the condition back to parallelogram, students can check whether this argument holds in the case of parallelogram. Then by backward/forward reasoning, they will confirm that triangles ABO and CBO nor ABO and ADO are not congruent, and based on the confirmation, they will also confirm that triangles ADO and CBO seem to be congruent.

On the case in which the conclusion does not hold

If students consider a trapezoid, by referring to a diagram, they can easily find that the conclusion does not generally hold. However, even in this case, based on the character (iii), they can look for what condition is necessary to be added, under which the conclusion OA = OC holds. By forward reasoning, the condition "ABCD is a trapezoid" and the property of parallel lines and alternate angles, they can find angle DAC = ACB and angle ADB = DBC (Figure 5).

By referring to Figure 5 and by backward reasoning, they can focus on a pair of triangle ADO and CBO whose two angles are equal. By backward reasoning, they can check the congruent conditions and will find out if AD = CB is added, based on ASA congruent condition, the conclusion OA = OC will hold. Based on the character (iii), they will make an argument in Figure 6, which is totally different from proof.

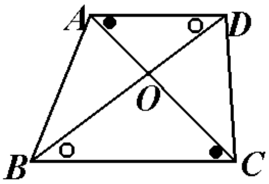

Figure 5 Diagram in the case of a trapezoid

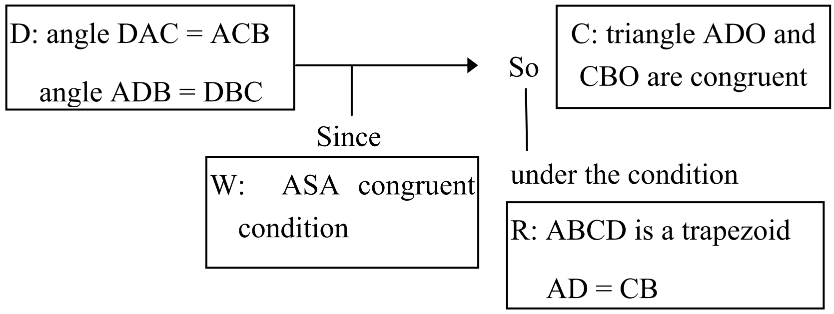

Figure 6 Argument in the case of a trapezoid

By changing the condition back to parallelogram, students will find the known statement "opposite sides in a parallelogram are equal". In this way, they can find out triangles ADO and CBO are congruent.

THE ROLE OF LOOKING BACK AT A PROCESS OF PROOF-PLANNING

Based on the proof-planning process illustrated above, we discuss the role of looking back at the process. First of all, in both cases, students manage to find a congruent pair of triangles ADO and CBO, which is a key for the proof. So it is confirmed that students possibly success in proof-planning by backward/forward reasoning, changing the condition and referring to diagrams. On the other hand, focus of this paper is on proof-planning processes which are not explicit in a proof. Note that arguments in Figure 4 and 6 are not explicit in proof as a product. As stated in the introduction, we discuss what kind of new problems students can find and how they can solve them.

The necessary condition of the statement

Firstly, we focus on the case of trapezoid, that is, the changed condition to the one in which the conclusion does not hold. By looking back at the proof-planning process, students can compare two cases of parallelogram and trapezoid, that is, when the conclusion holds and when it does not. By this comparison, they can find a question of what the necessary condition under which the conclusion holds.

The process related to Figure 6 shows that a key condition to implicate the conclusion OA = OC is AD = CB. In detail, in addition to angles DAC = ACB and angles ADB = DBC which also hold in the case of trapezoid, AD = CB holds under the condition of parallelogram, not trapezoid.

The value of this necessary condition is that students can obtain an answer to mathematically important question: "why, not in a trapezoid, but in a parallelogram, diagonals intersect at each midpoint". The answer was in the process. By looking back the process, they can find out: "because in parallelogram not only that a pair of opposite sides is parallel but also that these sides are equal".

This kind of understanding has been discussed in relation to functions of proof as a product (especially "explanation"). To utilize explanation from only product, students have to identify which condition is crucial by examining each deductive connection which appears in the proof. It seems difficult for many students. In contrast, the process illustrated above is experienced only for the purpose to prove the statement. By looking back the process, students will be able to obtain, namely, hidden by-products of proving.

A statement which was not proved

Second case is rhombus, when the conclusion holds. By looking back at the process related to Figure 4, students can find out a problem whether in rhombus ABCD triangles ABO and CBO are truly congruent, that is, a statement which they considered true but could not prove at that time. After proving statement α, they can use it to prove the above statement immediately.

This statement appears nowhere in the statement α or its proof at all. So, not proof as a product, but the very process of proving can bear such new properties.

CONCLUDING REMARKS

This paper examines the value of proving in school mathematics by focusing on proving processes which are not explicit in the product. The value of this aspect of proving has been hardly discussed in the literature. Although the related literature of Hanna & Barbeau (2008) discusses general methods or techniques useful for solving not only problem to prove but also problem to find, this paper focuses finding and solving problems related to the original problem. Nevertheless, this paper implicates the value of proving still different from what has been discussed as functions of proof.

ACKNOWLEDGEMENTS

This study is supported by research grants from the Japan Society for the Promotion of Science as a Research Fellow.

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